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INTEGRATION OF THE EQUATIONS OF PLANE STATIONARY NON-LINEAR FILTRATION*

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Plane stationary filtration of an incompressible liquid is considered. Special filtration laws are singled out, allowing efficient integration of a system of equations of motion which are linearized by a Legendre transformation and put into canonical form. (For Legendre transformations in gas dynamics see /1, 2/, and in non-linear filtration see /3/). The solution of the problem of a point source in a strip is given as an example for one of the suggested filtration laws with a limit slope.

1. Let $z = x + iy$ be the flow plane, v the magnitude of the filtration velocity vector, θ the angle of the filtration velocity vector with x-axis, $\varphi = -H + \text{const}$, where H is the pressure head, ψ the stream function and $\Phi(v)$ the function determining the filtration law. We have /4/

$$\varphi_x = \Phi(v)\cos\theta,\ \varphi_y = \Phi(v)\sin\theta,\ \psi_x = -v\sin\theta,\ \psi_y = v\cos\theta$$

(where the subscripts denote partial differentiation with respect to the corresponding variables).

We assume

$$S = -\varphi + x\varphi_x + y\varphi_y,\ T = -\psi + x\psi_x + y\psi_y$$

Taking v and θ as the independent variables, we obtain

$$S_v = \Phi'(v)X,\ S_\theta = \Phi(v)Y,\ T_v = Y,\ T_\theta = -vX$$
$$X = x\cos\theta + y\sin\theta,\ Y = y\cos\theta - x\sin\theta$$

These equalities give the linear system

$$S_\theta = \Phi(v)T_v,\ S_v = -\Phi'(v)v^{-1}T_\theta$$

and the transformation formula into the flow plane

$$z = e^{i\theta}(-v^{-1}T_\theta + i\Phi^{-1}S_\theta)$$

If, using the equations,

$$\chi = \sqrt{\frac{\Phi(v)\Phi'(v)}{v}},\quad \sigma = \int\sqrt{\frac{\Phi'(v)}{v\Phi(v)}}\,dv \tag{1.1}$$

we introduce a function χ and a variable σ to replace v, we arrive at a canonical system of equations that is well-known in gas dynamics:

$$S_\theta = \chi(\sigma)T_\sigma,\ S_\sigma = -\chi(\sigma)T_\theta$$

It is clear that the functions

$$\mu(\sigma, \theta) = S_\theta,\ \nu(\sigma, \theta) = T_\theta$$

satisfy the same system of equations as the functions $S(\sigma, \theta)$ and $T(\sigma, \theta)$. We have

$$\mu_\theta = \chi(\sigma)\nu_\sigma,\ \mu_\sigma = -\chi(\sigma)\nu_\theta \tag{1.2}$$

where the functions for which this system is written can be introduced directly via the equalities $\mu = \Phi Y$ and $\nu = -vX$. Using the functions $\mu(\sigma, \theta)$ and $\nu(\sigma, \theta)$, the transformation into the flow plane is given by the formula

$$z = e^{i\theta}(-\nu v^{-1} + i\mu\Phi^{-1}) \tag{1.3}$$

From (1.1) it follows that

$$\chi dv/d\sigma = \Phi,\ d\Phi/d\sigma = \chi v \tag{1.4}$$

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For a given function $\chi(\sigma)$ these relations can be considered as a system of ordinary differential equations, whose solution determines the functions $\Phi(\sigma; C_1, C_2)$ and $v(\sigma; C_1, C_2)$ with arbitrary constants C_1 and C_2. Hence for a given function $\chi(\sigma)$ a corresponding family of filtration laws is established in parameterized form (with parameter σ).

2. We will consider the cases when $\chi = a/\sigma^2$ and $\chi = a\,\mathrm{cth}^2\,\sigma$ $(a = \mathrm{const} > 0)$.
For the first case the solution of system (1.2) can be represented as

$$\mu = a\sigma^{-1}\,\mathrm{Re}\,F',\quad \nu = \mathrm{Im}\,(F - \sigma F') \tag{2.1}$$

and in the second case as

$$\mu = -a\,\mathrm{Re}\,(F - \mathrm{cth}\,\sigma F'),\quad \nu = \mathrm{Im}\,(F - \mathrm{th}\,\sigma F')$$

where F is an arbitrary analytic function of the complex variable $\omega = \sigma + i\theta$ /2/. Of course a representation also exists for the pair $S(\sigma, \theta)$ and $T(\sigma, \theta)$.
If $\chi(\sigma) = a/\sigma^2$ then the corresponding family of filtration laws is given by the formulae

$$\Phi = \sigma^{-1}(C_1\,\mathrm{sh}\,\sigma + C_2\,\mathrm{ch}\,\sigma),\quad v = a^{-1}[C_1(\sigma\,\mathrm{ch}\,\sigma - \mathrm{sh}\,\sigma) + C_2(\sigma\,\mathrm{sh}\,\sigma - \mathrm{ch}\,\sigma)]$$

We put $C_2 = 0$. We obtain a filtration law with a limit slope $(\lambda > 0$ and $\sigma \geqslant 0)$

$$\Phi/\lambda = \mathrm{sh}\,\sigma/\sigma,\quad av/\lambda = \sigma\,\mathrm{ch}\,\sigma - \mathrm{sh}\,\sigma \tag{2.2}$$

where instead of C_1 we have introduced the notation λ.

This law is shown in Fig.1 by curve *1*. Unlike previously proposed laws /3, 5, 6/ with limit slopes for the solution of plane filtration problems, this one is convex. The inequalities $d\Phi/dv > 0$ and $d^2\Phi/dv^2 < 0$ are satisfied for all v in the interval $[0, \infty)$. For $\sigma = 0$ we have $v = 0$, $\Phi = \lambda$, $d\Phi/dv = \infty$ and $d^2\Phi/dv^2 = -\infty$. If $\sigma \to \infty$ then $v \to \infty$, $\Phi \to \infty$, $d\Phi/dv \to 0$ and $d^2\Phi/dv^2 \to 0$.

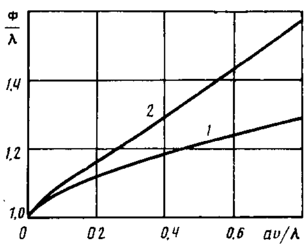

Fig.1

The family of filtration laws for $\chi(\sigma) = a\,\mathrm{cth}^2\,\sigma$ is given by the formulae

$$\Phi = \frac{C_1(\mathrm{sh}\,2\sigma + 2\sigma) - C_2}{\mathrm{sh}\,\sigma}$$
$$v = \frac{C_1(\mathrm{sh}\,2\sigma - 2\sigma) - C_2}{a\,\mathrm{ch}\,\sigma}$$

We put $C_1 = \lambda/4$ and $C_2 = 0$. We arrive at a filtration law with limit gradient $(\lambda > 0$, $\sigma \geqslant 0)$

$$\frac{\Phi}{\lambda} = \frac{\mathrm{sh}\,2\sigma + 2\sigma}{4\,\mathrm{sh}\,\sigma},\quad \frac{av}{\lambda} = \frac{\mathrm{sh}\,2\sigma - 2\sigma}{4\,\mathrm{ch}\,\sigma}$$

This law is represented by curve *2* in Fig.1. For small values of the filtration rate it has the property $d^2\Phi/dv^2 < 0$, (curve *2* on Fig.1 has a point of inflection). For all $v \geqslant 0$ we have $d\Phi/dv > 0$. For $\sigma = 0$ the equalities $v = 0$, $\Phi = \lambda$, $d\Phi/dv = \infty$, $d^2\Phi/dv^2 = -\infty$ are satisfied. If $\sigma \to \infty$, then $v \to \infty$, $\Phi \to \infty$, $d\Phi/dv \to a$ and $d^2\Phi/dv^2 \to 0$.

The simplest condition is $\chi = a = \mathrm{const}$. In this case $S - iaT$ and $\mu - ia\nu$ are analytic functions of ω. The corresponding family of filtration laws is given by the formulae

$$\Phi = C_1\,\mathrm{sh}\,\sigma + C_2\,\mathrm{ch}\,\sigma,\quad av = C_1\,\mathrm{ch}\,\sigma + C_2\,\mathrm{sh}\,\sigma$$

For $C_1 = C_2$ we have a linear law and for $C_1 = 0, a = 1$ the law considered in /5/.

3. We will apply law (2.2) to the problem of a point source in a strip.
Let the strip containing the flow be bounded by the lines $y = +l$ and $y = -l$. The walls bounding the flow are impermeable. A point source of strength $4q$ is located at the origin of coordinates. The magnitude of the filtration velocity vector at left and right infinity is $v_1 = q/l$, and we denote the corresponding value of σ by σ_1. By definition $v = 0$ at the boundaries of the stagnation zones and consequently $\sigma = 0$, $\nu = 0$.
By virtue of the flow's symmetry it is sufficient to investigate only its first quadrant. This part of the flow corresponds to the half-strip $0 < \theta < \pi/2$, $\sigma > 0$, in the ω plane, at whose boundary the equalities

$$\mu(\sigma, 0) = 0\ (\sigma > \sigma_1),\quad \mu(\sigma, 0) = l\lambda\sigma^{-1}\,\mathrm{sh}\,\sigma\ (0 < \sigma < \sigma_1)$$
$$\nu(0, \theta) = 0\ (0 \leqslant \theta \leqslant \pi/2),\quad \mu(\sigma, \pi/2) = 0\ (\sigma \geqslant 0)$$

are satisfied.
As a result of the substitution

$$F = \frac{1}{a} \int_{\omega_0}^{\omega} W(\omega)\, d\omega \quad (\omega_0 = \text{const})$$

we obtain in place of (2.1) the representation

$$\mu = \sigma^{-1} \operatorname{Re} W(\omega), \quad \nu = a^{-1} \operatorname{Im} \left[\int_{\omega_0}^{\omega} W(\omega)\, d\omega - \sigma W(\omega) \right] \tag{3.1}$$

with an arbitrary analytic function $W(\omega) = U(\sigma, \theta) + iV(\sigma, \theta)$.

Let $\omega_0 = i\pi/2$. The transformed boundary condition will be satisfied if the equality $U = l\lambda \operatorname{sh} \sigma$ is satisfied for $\theta = 0, 0 \leqslant \sigma < \sigma_1$, and on the remaining part of the boundary $U = 0$.

We map the half-strip under consideration in the ω plane onto the upper half-plane of the complex variable $\zeta = \xi + i\eta$ using the function $\zeta = \operatorname{ch} 2\omega$. On the boundary $\eta = 0$ we have

$$U = 0 \ (\xi \leqslant 1), \quad U = l\lambda \sqrt{(\xi - 1)/2} \quad (1 \leqslant \xi < \operatorname{ch} 2\sigma_1)$$
$$U = 0 \quad (\xi > \operatorname{ch} 2\sigma_1)$$

Having determined the function $W(\zeta)$ for the boundary data with the help of the Schwarz integral, we return to the variable ω and obtain the required function $W(\omega)$ in the form

$$W = \frac{l\lambda}{\pi i} \left(2 \operatorname{sh} \sigma_1 + \operatorname{sh} \omega \ln \frac{\operatorname{sh} \omega - \operatorname{sh} \sigma_1}{\operatorname{sh} \omega + \operatorname{sh} \sigma_1} \right) \tag{3.2}$$

The boundary of the stagnation zone is determined by the equation

$$z(\theta) = \lambda^{-1} e^{i\theta} \left[-U_{\sigma\theta} + iU_\sigma \right]_{\sigma=0}$$

which we obtain from (1.3) by applying L'Hopital's rule and taking into account the first of the formulae (1.2), (1.4) and (3.1) in the transformation process. Using solution (3.2) we establish the following formulae for constructing the boundary of the stagnation zone:

$$\frac{\pi x}{l} = \frac{4 \operatorname{sh}^3 \sigma_1 \cos^3 \theta}{(\sin^2 \theta + \operatorname{sh}^2 \sigma_1)^2}, \quad \frac{\pi y}{l} = \pi - 2 \operatorname{arctg} \frac{\sin \theta}{\operatorname{sh} \sigma_1} +$$
$$2 \operatorname{sh} \sigma_1 \sin \theta \ \frac{\operatorname{sh}^2 \sigma_1 \cos^2 \theta - \operatorname{ch}^2 \sigma_1 \sin^2 \theta}{(\sin^2 \theta + \operatorname{sh}^2 \sigma_1)^2}$$

The result of the calculation, carried out using these formulae for some values of the parameter $b = av_1/\lambda$, is shown in Fig.2.

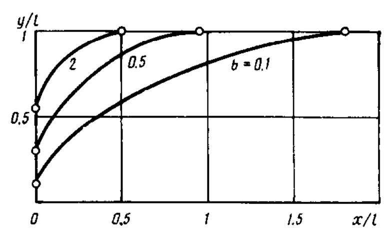

Fig.2

In the case $l = \infty$, $v_1 = 0$ we have the particularly simple solution

$$W = 2iaq/(\pi \operatorname{sh}^2 \omega)$$

This can be obtained by replacing l by q/v_1 in (3.2), using the second of formulae (2.2), and then taking the limit as $\sigma_1 \to 0$.

The boundary of the stagnation zone in the limiting case $l = \infty$, $v_1 = 0$ is given by the formula

$$z(\theta) = \frac{4aq}{\pi\lambda} \left(\frac{3 \cos^3 \theta}{\sin^4 \theta} + i \frac{1 + 3 \cos^2 \theta}{\sin^3 \theta} \right)$$

The result of a calculation using this formula is shown in Fig.3 in coordinates $x^* = \lambda x/(aq)$, $y^* = \lambda y/(aq)$.

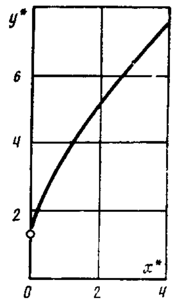

Fig.3

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